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ON GENERALIZED SHIFT BASES FOR THE WIENER DISC ALGEBRA

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ABSTRACT

Let $W(D)$ denote the set of functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$ for which $\sum_{n=0}^{\infty}|a_n|<+\infty$. Given any finite set $\{f_i(z)\}_{i=1}^{n}$ in $W(D)$ the following are equivalent: (i) The generalized shift sequence $\{f_1(z)z^{kn},f_2(z)z^{kn+1},\dots,f_n(z)z^{(k+1)n-1}\}_{k=0}^{\infty}$ is a basis for $W(D)$ which is equivalent to the basis $\{z^m\}_{m=0}^{\infty}$. (ii) The generalized shift sequence is complete in $W(D)$. (iii) The function

$$F(z)=\det\begin{bmatrix} f_1(z) & f_1(wz) & \cdots & f_1(w^{n-1}z)\\ f_2(z) & wf_2(wz) & \cdots & w^{n-1}f_2(w^{n-1}z)\\ \vdots & \vdots & & \vdots\\ f_n(z) & w^{n-1}f_n(wz) & \cdots & (w^{n-1})^{n-1}f_n(w^{n-1}z)\end{bmatrix}$$

has no zero in $|z|\leqq 1$, where $w=e^{2\pi i/n}$.

§1. Introduction

Let $W(D)$ denote the Wiener disc algebra consisting of all analytic functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$ for which $\|f\|=\sum_{n=0}^{\infty}|a_n|<+\infty$. Then $W(D)$ is isometrically isomorphic to the complex sequence space l^1 under the mapping $\sum_{n=0}^{\infty}a_nz^n\to(a_n)$ and the sequence $\{z^n\}_{n=0}^{\infty}$ is a Schauder basis for $W(D)$ which is equivalent to the unit vector basis $\{e_n\}_{n=1}^{\infty}$ for l^1. More generally it was shown in [2] that given any $f(z)$ in $W(D)$ the "shift sequence" $\{f(z)z^n\}_{n=0}^{\infty}$ is a basis for $W(D)$ if and only if $f(z)$ has no zero in the disc $D=\{z\mid |z|\leqq 1\}$, and that any such shift basis for $W(D)$ is equivalent to the standard basis $\{z^n\}_{n=0}^{\infty}$.

It is natural to inquire whether analogous results are valid for more general types of sequences of functions in $W(D)$. A first step in this direction was made in the paper [3] in which it was shown that for any integer $k\geqq 1$ and any function $f(z)$ in $W(D)$ the "multiply-shifted" sequence $\{f(z)z^{kn}\}_{n=0}^{\infty}$ is a basic sequence in $W(D)$ equivalent to $\{z^n\}_{n=0}^{\infty}$ if and only if $f(z)$ does not have k equally spaced

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zeros on the circle $|z| = 1$. In the present paper we take another direction which allows us to return to a consideration of bases for $W(D)$. In this extension of the notion of a shift sequence the single function which occurs in the definition of such a sequence is replaced by an arbitrary (finite) set of functions which is then permuted cyclically, thereby giving rise to what we call a *generalized shift sequence.*

DEFINITION. If $f_1(z), f_2(z), \dots, f_n(z)$ are functions in $W(D)$, the sequence

$$\{f_1(z), f_2(z)z, \dots, f_n(z)z^{n-1}, \dots, f_1(z)z^{kn}, \dots, f_n(z)z^{(k+1)n-1}, \dots\}$$

is called a *generalized shift sequence* (abbreviated g.s.s.) and denoted by $\{f_1(z)z^{kn}, \dots, f_n(z)z^{(k+1)n-1}\}_{k=0}^{\infty}$.

The purpose of this paper is not only to describe the basis behavior of such g.s.s., but to demonstrate the very interesting way in which function theory enters into this description (just as in [2] and [3]). We associate with the set $\{f_1(z), f_2(z), \dots, f_n(z)\}$ in $W(D)$ a function $F(z)$ in $W(D)$ in such a way that the g.s.s. determined by the set $\{f_1(z), \dots, f_n(z)\}$ is a basis if and only if $F(z)$ has no zero in D. Once this has been done the Wiener–Levy theorem can be applied as in [2] to get the equivalence of the basis to the basis $\{z^n\}_{n=0}^{\infty}$ for $W(D)$. The details of this relationship are given in the following theorem (and its proof).

THEOREM. *If $\{f_1(z), f_2(z), \dots, f_n(z)\}$ is a set of functions in $W(D)$, the following are equivalent:*

(i) *The g.s.s. $\{f_1(z)z^{kn}, \dots, f_n(z)z^{(k+1)n-1}\}_{k=0}^{\infty}$ is a basis for $W(D)$ which is equivalent to the basis $\{z^m\}_{m=0}^{\infty}$.*

(ii) *The g.s.s. $\{f_1(z)z^{kn}, \dots, f_n(z)z^{(k+1)n-1}\}_{k=0}^{\infty}$ is complete in $W(D)$ (i.e., the linear span of the sequence is dense in $W(D)$).*

(iii) *If $w = e^{2\pi i/n}$, the function*

$$F(z) = \det \begin{bmatrix} f_1(z) & f_1(wz) & \cdots & f_1(w^r z) & \cdots & f_1(w^{n-1}z) \\ \vdots & \vdots & & & & \vdots \\ f_k(z) & w^{k-1}f_k(wz) & \cdots & (w^{k-1})^r f_k(w^r z) & \cdots & (w^{k-1})^{n-1} f_k(w^{n-1}z) \\ \vdots & \vdots & & & & \vdots \\ f_n(z) & w^{n-1}f_n(wz) & \cdots & (w^{n-1})^r f_n(w^r z) & \cdots & (w^{n-1})^{n-1} f_n(w^{n-1}z) \end{bmatrix}$$

has no zero in D.

A comparison of this result with the special case ([2], Theorem 4.1) of an ordinary shift basis in $W(D)$ shows the rather remarkable extent to which the methods of [2] can be applied in a more general setting. Moreover, it suggests that other problems concerning g.s.s. (e.g., characterization of basic sequences and their equivalences) may be treated by applying the methods of [2] to the sequence $\{F(z)z^n\}_{n=0}^{\infty}$. We return briefly to a discussion of this possibility in §3.

§2. Proof of the Theorem

In order to show that (i), (ii) and (iii) are equivalent we prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

The fact that (i) $\Rightarrow$ (ii) is an obvious consequence of the definition of a basis.

(ii) $\Rightarrow$ (iii): For each $i = 1,2,\dots,n$, let $f_i(z) = \sum_{m=0}^{\infty} a_m^{(i)} z^m$. Since $W(D)$ is isometrically isomorphic to l^1 under the mapping $\sum_{n=0}^{\infty} a_n z^n \Leftrightarrow \{a_n\}_{n=0}^{\infty}$, the sequence $\{f_1(z)z^{kn},\dots,f_n(z)z^{(k+1)n-1}\}_{k=0}^{\infty}$ can be identified with the sequence $\{x_j\}_{j=1}^{\infty}$ in l^1 defined by

$$x_1 = (a_0^{(1)}, a_1^{(1)}, a_2^{(1)}, a_3^{(1)},\dots),$$

$$x_2 = (0, a_0^{(2)}, a_1^{(2)}, a_2^{(2)},\dots),$$

$$x_3 = (0,0, a_0^{(3)}, a_1^{(3)}, a_2^{(3)},\dots),$$

$$\vdots$$

$$x_n = (0,0,\dots,0, a_0^{(n)}, a_1^{(n)},\dots),$$

and $x_{kn+i} = S^{kn}(x_i)$ for all $k \geqq 1$ and $1 \leqq i \leqq n$, where

$$S^p(c_1,c_2,\dots) = (\underbrace{0,0,\dots,0}_{p \text{ places}}, c_1, c_2,\dots)$$

for $p \geqq 1$ and (c_i) in l^1.

By assumption $\{x_j\}_{j=1}^{\infty}$ is complete in l^1. Hence $a_0^{(i)} \neq 0$, and $f_i(0) \neq 0$, for $i = 1,2,\dots,n$.

Let

$$A(z) = \begin{bmatrix} f_1(z) & f_1(wz) & \cdots & f_1(w^{n-1}z) \\ \vdots & \vdots & & \vdots \\ f_k(z) & w^{k-1}f_k(wz) & \cdots & (w^{k-1})^{n-1}f_k(w^{n-1}z) \\ \vdots & \vdots & & \vdots \\ f_n(z) & & \cdots & (w^{n-1})^{n-1}f_n(w^{n-1}z) \end{bmatrix},$$

so $F(z) = \det A(z)$.

If $F(0) = 0$, then expanding $\det A(z)$ gives

$$0 = F(0) = f_1(0)\cdots f_n(0)\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^{n-1} \\ \vdots & & & \vdots \\ 1 & w^{n-1} & \cdots & (w^{n-1})^{n-1} \end{bmatrix},$$

an impossibility since $f_i(0)\neq 0$ for all i and Vandermonde's determinant is not zero for w a primitive nth root of unity.

Thus if $F(z_0)=0$ for some z_0 in D then $0<|z_0|\leqq 1$. In this case the homogeneous system $A(z_0)\mathbf{x}=\mathbf{0}$ has a non-trivial solution, say $\mathbf{x}=(c_1,c_2,\dots,c_n)$. If we define a set of functionals $\{h_i\}_{i=1}^n$ in l^∞ by

$$h_i=(1,w^{i-1}z_0,(w^{i-1}z_0)^2,\dots,(w^{i-1}z_0)^k,\dots)$$

then, just as above, using the fact that $z_0\neq 0$ and properties of Vandermonde's determinant we conclude $\{h_i\}_{i=1}^n$ is a linearly independent set and hence that $H=\Sigma_{i=1}^n c_ih_i\neq 0$. However, if $1\leqq j\leqq n$ then

$$\begin{aligned}\langle H,x_j\rangle=\sum_{i=1}^n c_i\langle h_i,x_j\rangle=\sum_{i=1}^n c_i(w^{i-1}z_0)^{j-1}f_j(w^{i-1}z_0)&=z_0^{j-1}\sum_{i=1}^n c_i(w^{j-1})^{i-1}f_j(w^{i-1}z_0)\\&=z_0^{j-1}\cdot 0=0\end{aligned}$$

(by definition of $\{c_i\}_{i=1}^n$), while if $j>n$ then $j=mn+k$ for some $m\geqq 1$ and $1\leqq k\leqq n$ so again $\langle H,x_j\rangle=z_0^{mn}\langle H,x_k\rangle=0$ (by the above). That is, H is a non-zero annihilator of $\{x_j\}_{j=1}^\infty$, a contradiction to the completeness of $\{x_j\}_{j=1}^\infty$, so it must be that $F(z_0)\neq 0$ for all $z_0\in D$ and (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i): Suppose $F(z)\neq 0$ for z in D. Then the matrix $A(z)$ defined above is invertible for all z in D, as is its transpose $A^t(z)$. Moreover

$$[A^t(z)]^{-1}=[A^{-1}(z)]^t=\left[\frac{\operatorname{cof}A^t(z)}{\det A(z)}\right]^t=\frac{\operatorname{cof}A(z)}{\det A(z)}.$$

If we define a set of functions $\{F_j(z)\}_{j=1}^n$ by

$$\begin{bmatrix}F_1(z)\\F_2(z)\\\vdots\\F_n(z)\end{bmatrix}=[A^t(z)]^{-1}\cdot\begin{bmatrix}F(z)\\\vdots\\F(z)\end{bmatrix}=\frac{\operatorname{cof}A(z)}{\det A(z)}\cdot\begin{bmatrix}F(z)\\\vdots\\F(z)\end{bmatrix}$$

then we have (since $F(z)=\det A(z)$) that $F_j(z)=[C_{j1}(z)+\cdots+C_{jn}(z)]$, where $C_{jk}(z)$ is the cofactor of the (j,k)-entry in $A(z)$, or, equivalently,

$$F_j(z)=\det\begin{bmatrix}f_1(z)&f_1(wz)&\cdots&f_1(w^{n-1}z)\\\vdots&\vdots&&\vdots\\f_{j-1}(z)&(w^{j-2})f_{j-1}(wz)&\cdots&(w^{j-2})^{n-1}f_{j-1}(w^{n-1}z)\\1&1&\cdots&1\\f_{j+1}(z)&(w^j)f_{j+1}(wz)&\cdots&(w^j)^{n-1}f_{j+1}(w^{n-1}z)\\\vdots&\vdots&&\vdots\\f_n(z)&(w^{n-1})f_n(w^{n-1}z)&\cdots&(w^{n-1})^{n-1}f_n(w^{n-1}z)\end{bmatrix}.$$

Now

$$\begin{bmatrix} F(z) \\ \vdots \\ F(z) \end{bmatrix} = A^t(z) \cdot \begin{bmatrix} F_1(z) \\ \vdots \\ F_n(z) \end{bmatrix},$$

so by definition of $A^t(z)$ we have $F(z) = \sum_{j=1}^n F_j(z) f_j(z)$. We are assuming $F(z) \neq 0$ for z in D so by the Wiener–Levy theorem [1, p. 97] the function $F^{-1}(z)$ is in $W(D)$ and we have

$$1 = \sum_{j=1}^n \frac{F_j(z)}{F(z)} f_j(z) = \sum_{j=1}^n G_j(z) f_j(z),$$

where $G_j(z) = F_j(z)/F(z)$ is in $W(D)$ for all $j = 1, 2, \ldots, n$.

CLAIM. $G_j(wz) = w^{j-1} G_j(z)$, $j = 1, 2, \ldots, n$.

Clearly it is sufficient to show

$$F_j(wz) = w^{j-1} F_j(z) \quad \text{and} \quad F(wz) = F(z).$$

According to the expression we derived earlier for $F_j(z)$ we have

$$F_j(wz) = \det \begin{bmatrix} f_1(wz) & f_1(w^2z) & \cdots & f_1(z) \\ \vdots & \vdots & & \vdots \\ f_{j-1}(wz) & (w^{j-2}) f_{j-1}(w^2z) & \cdots & (w^{j-2})^{n-1} f_{j-1}(z) \\ 1 & 1 & \cdots & 1 \\ f_{j+1}(wz) & (w^j) f_{j+1}(w^2z) & \cdots & (w^j)^{n-1} f_{j+1}(z) \\ \vdots & \vdots & & \vdots \\ f_n(wz) & (w^{n-1}) f_n(w^2z) & \cdots & (w^{n-1})^{n-1} f_n(z) \end{bmatrix}.$$

Shifting columns to the right this becomes:

$$F_j(wz) = (-1)^{n-1} \det \begin{bmatrix} f_1(z) & f_1(wz) & \cdots & f_1(w^{n-1}z) \\ \vdots & \vdots & & \vdots \\ (w^{j-2})^{n-1} f_{j-1}(z) & f_{j-1}(wz) & \cdots & (w^{j-2})^{n-2} f_{j-1}(w^{n-1}z) \\ 1 & 1 & \cdots & 1 \\ (w^j)^{n-1} f_{j+1}(z) & f_{j+1}(wz) & \cdots & (w^j)^{n-2} f_{j+1}(w^{n-1}z) \\ \vdots & \vdots & & \vdots \\ (w^{n-1})^{n-1} f_n(z) & f_n(wz) & \cdots & (w^{n-1})^{n-2} f_n(w^{n-1}z) \end{bmatrix}$$

Now using the fact that $w^{n-1} = 1/w$, factor $(1/w)^{n-1}$ out of the nth row, $(1/w)^j$ out of the $(j+1)$st row, etc., and note that the above then becomes

$$F_j(wz)=\left[(-1)^{n-1}\left(\frac{1}{w^{n-1}}\right)\left(\frac{1}{w^{n-2}}\right)\cdots\left(\frac{1}{w^{j}}\right)\left(\frac{1}{w^{j-2}}\right)\cdots\left(\frac{1}{w}\right)\right]F_j(z)$$

$$=w^{j-1}F_j(z)\frac{(-1)^{n-1}}{\prod_{k=1}^{n-1}w^k}=w^{j-1}F_j(z).$$

By the same argument one shows that $F(wz)=F(z)$, and it follows that $G_j(z)=F_j(z)/F(z)$ has the property that $G_j(wz)=w^{j-1}G_j(z)$.

Now it is trivial to verify that if $h(z)$ is any function in $W(D)$ for which $h(wz)=w^{j-1}h(z)$ (for some $1\leqq j\leqq n$), then $h(z)=\Sigma_{k=0}^{\infty}b_kz^{kn+(j-1)}$ for some sequence $\{b_k\}_{k=0}^{\infty}$ in l^1. Therefore for each $j=1,2,\dots,n$ there is a sequence $\{b_k^{(j)}\}_{k=0}^{\infty}$ in l^1 for which $G_j(z)=\Sigma_{k=0}^{\infty}b_k^{(j)}z^{kn+(j-1)}$, and hence for which (by the above)

$$1=\sum_{j=1}^{n}G_j(z)f_j(z)=\sum_{j=1}^{n}\left[\sum_{k=0}^{\infty}b_k^{(j)}z^{kn+(j-1)}\right]f_j(z)=\sum_{k=0}^{\infty}\sum_{j=1}^{n}b_k^{(j)}f_j(z)z^{nk+(j-1)}.$$

That is, the function 1 has an expansion in terms of the g.s.s. $\{f_1(z)z^{kn},\dots,f_n(z)z^{(k+1)n-1}\}_{k=0}^{\infty}$ in which the coefficients are in l^1.

By the argument used to show that $G_j(wz)=w^{j-1}G(z)$ one can show that $F(z)$ can be alternately written as

$$F(z)=\det\begin{bmatrix} f_2(z) & f_2(wz) & \cdots & f_2(w^{n-1}z)\\ f_3(z) & wf_3(wz) & \cdots & w^{n-1}f_3(w^{n-1}z)\\ \vdots & \vdots & & \vdots\\ f_n(z) & w^{n-2}f_n(wz) & \cdots & (w^{n-2})^{n-1}f_n(w^{n-1}z)\\ f_1(z) & w^{n-1}f_1(wz) & \cdots & (w^{n-1})^{n-1}f_1(w^{n-1}z)\end{bmatrix},$$

and hence that $1=R_1(z)f_2(z)+\cdots+R_{n-1}(z)f_n(z)+R_n(z)f_1(z)$, where $R_j(z)$ is in $W(D)$ for $j=1,2,\dots,n$ and $R_j(wz)=w^{j-1}R_j(z)$. If we set $f_{n+1}(z)=f_1(z)$ then by the same argument as above we have

$$1=\sum_{k=0}^{\infty}\sum_{j=1}^{n}c_k^{(j)}f_{j+1}(z)z^{nk+(j-1)},$$

where $\{c_k^{(j)}\}$ is in l^1. It follows that

$$z=\sum_{k=0}^{\infty}\sum_{j=1}^{n}c_k^{(j)}f_{j+1}(z)z^{nk+j},$$

an expansion of the function z in terms of the g.s.s. which has coefficients in l^1.

In the same way, for each $j=0,1,\dots,n-1$ we obtain an expansion of the function z^j which has the form

$$z^j = p_j^{(j)} f_{j+1}(z) z^j + \cdots + p_{n-1}^{(j)} f_n(z) z^{n-1} + p_n^{(j)} f_1(z) z^n + p_{n+1}^{(j)} f_2(z) z^{n+1} + \cdots,$$

where $\{p_k^{(j)}\}_{k=0}^{\infty}$ is in l^1 for each $j = 0, 1, \ldots, n-1$. Thus if m is any integer $\geqq 0$ we have $m = rn + j$ for some $r \geqq 0$ and $0 \leqq j \leqq n-1$, so

$$z^m = z^{rn+j} = z^{rn} \cdot z^j = p_j^{(j)} f_{j+1}(z) z^m + p_{j+1}^{(j)} f_{j+2}(z) z^{m+1} + \cdots,$$

an expansion of z^m in terms of the g.s.s. for which the coefficients are always one of the n sequences $\{p_k^{(j)}\}_{k=0}^{\infty}$, $j = 0, 1, 2, \ldots, n-1$, in l^1.

In other words, if we denote the g.s.s. $\{f_1(z), f_2(z)z, \ldots, f_n(z)z^{n-1}, f_1(z)z^n, \ldots\}$ by $\{g_i(z)\}_{i=0}^{\infty}$ then for every integer $m = 0, 1, 2, \ldots$ there is a sequence $\{d_{mi}\}_{i=0}^{\infty}$ in l^1 for which

(1) $z^m = \sum_{i=0}^{\infty} d_{mi} g_i(z)$, and

(2) $\sup_m \sum_{i=0}^{\infty} |d_{mi}| < +\infty$.

Now let $k_1 = \sup_i \|g_i\| = \max_{1 \leqq j \leqq n} \|f_j\|$, and let $k_2 = \sup_m \sum_{i=0}^{\infty} |d_{m,i}|$. If $f(z) = \sum_{m=0}^{\infty} c_m z^m$ is in $W(D)$, then $\sum_{m,i=0}^{\infty} \|c_m d_{m,i} g_i\| \leqq k_1 k_2 \|f\|$. Therefore $f(z) = \sum_{i=0}^{\infty} b_i g_i(z)$ with $b_i = \sum_{m=0}^{\infty} c_m d_{m,i}$, and $k_1^{-1}\|f\| \leqq \sum_{i=0}^{\infty} |b_i| \leqq k_2 \|f\|$. It follows that $\{g_i\}$ is a basis for $W(D)$ which is equivalent to the basis $\{z^i\}_{i=0}^{\infty}$, and the proof is complete.

§3. Remarks

The theorem just proved gives a criterion for a g.s.s. to be a basis for $W(D)$, namely that the associated function $F(z)$ have no zero on the disc D. A consequence of this result and that of [2] is that the g.s.s. is a basis for $W(D)$ if and only if the sequence $\{F(z)z^n\}_{n=0}^{\infty}$ is a basis for $W(D)$. Moreover, in this case the two bases are equivalent to one another, since each is equivalent to the basis $\{z^n\}_{n=0}^{\infty}$ for $W(D)$.

A natural question is whether analogous criteria can be developed in studying the more general question of which g.s.s. are *basic sequences* in $W(D)$ (i.e., bases for their closed linear span). At this time the possibility of an answer to the question seems to be rather remote. Even in the case of an ordinary shift sequence $\{f(z)z^n\}_{n=0}^{\infty}$, no characterization of its basic sequence behavior is known in the general case. However when $f(z)$ is a polynomial a complete characterization has been given in [2], once again in terms of the zeros of $f(z)$ in D. Thus one would hope to be able to study the basic sequence hehavior of generalized shift sequences of *polynomials* in terms of the basic sequence behavior of $\{F(z)z^n\}_{n=0}^{\infty}$ (which is known [2], since in this case $F(z)$ is a polynomial).

In this setting preliminary results are mixed and not in a definitive form. For example, one can show that if $f(z)$ and $g(z)$ are polynomials then the g.s.s. $\{f(z)z^{2n}, g(z)z^{2n+1}\}_{n=0}^{\infty}$ is a basic sequence in $W(D)$ if and only if the associated shift sequence $\{F(z)z^n\}_{n=0}^{\infty}$ is a basic sequence in $W(D)$. However (in contrast to the basis result proved above) in this case the two basic sequences need not be equivalent. Thus it appears that generalized shift basic sequences are considerably more difficult to study from this standpoint than are generalized shift bases for $W(D)$. In a subsequent paper we will give the details of the above-mentioned results as well as others which extend the relationship between a g.s.s. and its associated shift sequence $\{F(z)z^n\}_{n=0}^{\infty}$ which has been introduced here.

References

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